Bosonic String and String Field Theory: a Solution Using Ultradistributions of Exponential Type

C.G. Bollini · M.C. Rocca

Abstract In this paper we show that Ultradistributions of Exponential Type (UET) are appropriate for the description in a consistent way string and string field theories. A new Lagrangian for the closed string is obtained and shown to be equivalent to Nambu–Goto's Lagrangian. We also show that the string field is a linear superposition of UET of compact support (CUET). We evaluate the propagator for the string field, and calculate the convolution of two of them.

Keywords Ultradistributions · Fields · Strings

1 Introduction

In a series of papers [1–5] we have shown that Ultradistribution theory of Sebastiao e Silva [6–8] permits a significant advance in the treatment of quantum field theory. In particular, with the use of the convolution of Ultradistributions we have shown that it is possible to define a general product of distributions (a product in a ring with divisors of zero) that sheds new ligth on the question of the divergences in Quantum Field Theory. Furthermore, Ultradistributions of Exponential Type (UET) are adequates to describe Gamow States and exponentially increasing fields in Quantum Field Theory [9–11].

Ultradistributions also have the advantage of being representable by means of analytic functions. So that, in general, they are easier to work with and, as we shall see, have interesting properties. One of those properties is that Schwartz's tempered distributions are canonical and continuously injected into Ultradistributions of Exponential Type and as a consequence the Rigged Hilbert Space with tempered distributions is canonical and continuously included in the Rigged Hilbert Space with Ultradistributions of Exponential Type.

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Another interesting property is that the space of UET is reflexive under the operation of Fourier transform (in a similar way of tempered distributions of Schwartz).

In this paper we show that Ultradistributions of Exponential type provides an adequate framework for a consistent treatment of string and string field theories. In particular, a general state of the closed bosonic bradyonic string is represented by UET of compact support, and as a consequence the string field of a bradyonic bosonic string is a linear combination of UET of compact support (CUET).

This paper is organized as follows: in Sects. 2 and 3 we define the Ultradistributions of Exponential Type and their Fourier transform. They are part of a Guelfand's Triplet (or Rigged Hilbert Space [12]) together with their respective duals and a "middle term" Hilbert space. In Sect. 4 we treat the question of the equivalence of Nanbu-Goto Lagrangian with a new Lagrangian for the closed bradyonic bosonic string. In Sect. 5 we obtain a expression for the Lagrangian of a closed tachyonic bosonic string. In Sect. 6 we give a expression for the propagator of the string (do not confuse with the string field propagator of Sect. 9). In Sect. 7 we give a new representation for the states of the string using CUET of compact support. In Sect. 8 we give expressions for the field of the string, the string field propagator and the creation and annihilation operators of a string. In Sect. 9, we give expressions for the non-local action of a free string and a non-local interaction Lagrangian for the string field similar to $\lambda\phi^4$ in Quantum Field Theory. Also we show how to evaluate the convolution of two string field propagators. Finally, Sect. 10 is reserved for a discussion of the principal results.

2 Ultradistributions of Exponential Type

Let $\mathcal{S}$ be the Schwartz space of rapidly decreasing test functions. Let Λ_j be the region of the complex plane defined as:

$$\Lambda_j = \{z \in \mathbb{C} : |\Im(z)| < j : j \in \mathbb{N}\}. \tag{2.1}$$

According to [6, 8] be the space of test functions $\hat{\phi} \in \mathfrak{V}_j$ is constituted by all entire analytic functions of $\mathcal{S}$ for which

$$\|\hat{\phi}\|_j = \max_{k \leq j}\{\sup_{z \in \Lambda_j}[e^{(j|\Re(z)|)}|\hat{\phi}^{(k)}(z)|]\} \tag{2.2}$$

is finite.

The space $\mathfrak{Z}$ is then defined as:

$$\mathfrak{Z} = \bigcap_{j=0}^{\infty} \mathfrak{V}_j. \tag{2.3}$$

It is a complete countably normed space with the topology generated by the system of seminorms $\{\|\cdot\|_j\}_{j\in\mathbb{N}}$. The dual of $\mathfrak{Z}$, denoted by $\mathfrak{B}$, is by definition the space of ultradistributions of exponential type [6, 8]. Let $\mathfrak{S}$ be the space of rapidly decreasing sequences. According to [12] $\mathfrak{S}$ is a nuclear space. We consider now the space of sequences $\mathfrak{P}$ generated by the Taylor development of $\hat{\phi} \in \mathfrak{Z}$:

$$\mathfrak{P} = \left\{\mathfrak{Q} : \mathfrak{Q} = \left(\hat{\phi}(0), \hat{\phi}'(0), \frac{\hat{\phi}''(0)}{2}, \ldots, \frac{\hat{\phi}^{(n)}(0)}{n!}, \ldots\right) : \hat{\phi} \in \mathfrak{Z}\right\}. \tag{2.4}$$

The norms that define the topology of $\mathfrak{P}$ are given by:

$$\|\hat{\phi}\|_p' = \sup_n \frac{n^p}{n} |\hat{\phi}^n(0)|. \tag{2.5}$$

$\mathfrak{P}$ is a subspace of $\mathfrak{S}$ and therefore is a nuclear space. As the norms $\| \cdot \|_j$ and $\| \cdot \|_p'$ are equivalent, the correspondence

$$\mathfrak{Z} \Longleftrightarrow \mathfrak{P} \tag{2.6}$$

is an isomorphism and therefore $\mathfrak{Z}$ is a countably normed nuclear space. We can define now the set of scalar products

$$\langle \hat{\phi}(z), \hat{\psi}(z) \rangle_n = \sum_{q=0}^n \int_{-\infty}^\infty e^{2n|z|} \overline{\hat{\phi}^{(q)}}(z) \hat{\psi}^{(q)}(z) dz = \sum_{q=0}^n \int_{-\infty}^\infty e^{2n|x|} \overline{\hat{\phi}^{(q)}}(x) \hat{\psi}^{(q)}(x) dx. \tag{2.7}$$

This scalar product induces the norm

$$\|\hat{\phi}\|_n'' = [\langle \hat{\phi}(x), \hat{\phi}(x) \rangle_n]^{\frac{1}{2}}. \tag{2.8}$$

The norms $\| \cdot \|_j$ and $\| \cdot \|_n''$ are equivalent, and therefore $\mathfrak{Z}$ is a countably hilbertian nuclear space. Thus, if we call now $\mathfrak{Z}_p$ the completion of $\mathfrak{Z}$ by the norm p given in (2.8), we have:

$$\mathfrak{Z} = \bigcap_{p=0}^\infty \mathfrak{Z}_p, \tag{2.9}$$

where

$$\mathfrak{Z}_0 = \mathbf{H} \tag{2.10}$$

is the Hilbert space of square integrable functions.

As a consequence the "nested space"

$$\mathfrak{U} = (\mathfrak{Z}, \mathbf{H}, \mathfrak{B}) \tag{2.11}$$

is a Guelfand's triplet (or a Rigged Hilbert Space=RHS. See [12]).

Any Guelfand's triplet $\mathfrak{G} = (\mathbf{\Phi}, \mathbf{H}, \mathbf{\Phi}')$ has the fundamental property that a linear and symmetric operator on $\mathbf{\Phi}$, admitting an extension to a self-adjoint operator in $\mathbf{H}$, has a complete set of generalized eigen-functions in $\mathbf{\Phi}'$ with real eigenvalues.

$\mathfrak{B}$ can also be characterized in the following way [6, 8]: let $\mathfrak{E}_\omega$ be the space of all functions $\hat{F}(z)$ such that:

I $\hat{F}(z)$ is analytic for $\{z \in \mathbb{C} : |\text{Im}(z)| > p\}$.
II $\hat{F}(z) e^{-p|\Re(z)|}/z^p$ is bounded continuous in $\{z \in \mathbb{C} : |\text{Im}(z)| \geqq p\}$, where $p = 0, 1, 2, \dots$ depends on $\hat{F}(z)$.

Let $\mathfrak{N}$ be: $\mathfrak{N} = \{\hat{F}(z) \in \mathfrak{E}_\omega : \hat{F}(z) \text{ is entire analytic}\}$. Then $\mathfrak{B}$ is the quotient space:

III $\mathfrak{B} = \mathfrak{E}_\omega / \mathfrak{N}$.

Due to these properties it is possible to represent any ultradistribution as [6, 8]:

$$\hat{F}(\hat{\phi}) = \langle \hat{F}(z), \hat{\phi}(z) \rangle = \oint_\Gamma \hat{F}(z) \hat{\phi}(z) dz, \tag{2.12}$$

where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $\text{Im}(z) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $\text{Im}(z) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $\hat{F}(z)$.)

Formula (2.12) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of "Dirac formula" for exponential ultradistributions [6]:

$$\hat{F}(z) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{t-z} dt \equiv \frac{\cosh(\lambda z)}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t-z)\cosh(\lambda t)} dt, \tag{2.13}$$

where the "density" $\hat{f}(t)$ is such that

$$\oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) dz = \int_{-\infty}^{\infty} \hat{f}(t)\hat{\phi}(t) dt \tag{2.14}$$

(2.13) should be used carefully. While $\hat{F}(z)$ is analytic on Γ, the density $\hat{f}(t)$ is in general singular, so that the r.h.s. of (2.14) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $\hat{F}(z)$ is bounded by a exponential and a power of z [6, 8]:

$$|\hat{F}(z)| \leq C|z|^p e^{p|\Re(z)|}, \tag{2.15}$$

where C and p depend on $\hat{F}$.

The representation (2.12) implies that the addition of any entire function $\hat{G}(z) \in \mathfrak{N}$ to $\hat{F}(z)$ does not alter the ultradistribution:

$$\oint_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z) dz = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) dz + \oint_{\Gamma} \hat{G}(z)\hat{\phi}(z) dz.$$

But:

$$\oint_{\Gamma} \hat{G}(z)\hat{\phi}(z) dz = 0$$

as $\hat{G}(z)\hat{\phi}(z)$ is entire analytic (and rapidly decreasing),

$$\therefore \quad \oint_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z) dz = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) dz. \tag{2.16}$$

Another very important property of $\mathfrak{B}$ is that $\mathfrak{B}$ is reflexive under the Fourier transform:

$$\mathfrak{B} = \mathcal{F}_c\{\mathfrak{B}\} = \mathcal{F}\{\mathfrak{B}\}, \tag{2.17}$$

where the complex Fourier transform $F(k)$ of $\hat{F}(z) \in \mathfrak{B}$ is given by:

$$F(k) = \Theta[\Im(k)] \int_{\Gamma_+} \hat{F}(z)e^{ikz} dz - \Theta[-\Im(k)] \int_{\Gamma_-} \hat{F}(z)e^{ikz} dz$$

$$= \Theta[\Im(k)] \int_{0}^{\infty} \hat{f}(x)e^{ikx} dx - \Theta[-\Im(k)] \int_{-\infty}^{0} \hat{f}(x)e^{ikx} dx. \tag{2.18}$$

Here Γ_+ is the part of Γ with $\Re(z) \geq 0$ and Γ_- is the part of Γ with $\Re(z) \leq 0$. Using (2.18) we can interpret Dirac's formula as:

$$F(k) \equiv \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f(s)}{s-k} ds \equiv \mathcal{F}_c\{\mathcal{F}^{-1}\{f(s)\}\}. \tag{2.19}$$

The treatment for ultradistributions of exponential type defined on $\mathbb{C}^n$ is similar to the case of one variable. Thus

$$\Lambda_j = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : |\Im(z_k)| \leq j, \ 1 \leq k \leq n\}, \tag{2.20}$$

$$\|\hat{\phi}\|_j = \max_{k \leq j}\left\{\sup_{z \in \Lambda_j}\left[e^{j[\sum_{p=1}^{n} |\Re(z_p)|]}|D^{(k)}\hat{\phi}(z)|\right]\right\}, \tag{2.21}$$

where $D^{(k)} = \partial^{(k_1)}\partial^{(k_2)}\cdots\partial^{(k_n)}k = k_1 + k_2 + \cdots + k_n$.

$\mathfrak{B}^n$ is characterized as follows. Let $\mathfrak{E}_\omega^n$ be the space of all functions $\hat{F}(z)$ such that:

I $\hat{F}(z)$ is analytic for $\{z \in \mathbb{C}^n : |\mathrm{Im}(z_1)| > p, |\mathrm{Im}(z_2)| > p, \ldots, |\mathrm{Im}(z_n)| > p\}$.

II $\hat{F}(z)e^{-[p\sum_{j=1}^{n} |\Re(z_j)|]}/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |\mathrm{Im}(z_1)| \geqq p, |\mathrm{Im}(z_2)| \geqq p, \ldots, |\mathrm{Im}(z_n)| \geqq p\}$, where $p = 0, 1, 2, \ldots$ depends on $\hat{F}(z)$.

Let $\mathfrak{N}^n$ be: $\mathfrak{N}^n = \{\hat{F}(z) \in \mathfrak{E}_\omega^n : \hat{F}(z)$ is entire analytic at minus in one of the variables $z_j, \ 1 \leq j \leq n\}$. Then $\mathfrak{B}^n$ is the quotient space:

III $\mathfrak{B}^n = \mathfrak{E}_\omega^n/\mathfrak{N}^n$. We have now

$$\hat{F}(\hat{\phi}) = \langle\hat{F}(z), \hat{\phi}(z)\rangle = \oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)dz_1 dz_2 \cdots dz_n. \tag{2.22}$$

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \cdots \Gamma_n$, where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $\mathrm{Im}(z_j) > \zeta, \ \zeta > p$ and back from ∞ to $-\infty$ for $\mathrm{Im}(z_j) < -\zeta, \ -\zeta < -p$. (Again Γ surrounds all the singularities of $\hat{F}(z)$.) The n-dimensional Dirac's formula is

$$\hat{F}(z) = \frac{1}{(2\pi i)^n} \int\limits_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t_1 - z_1)(t_2 - z_2)\cdots(t_n - z_n)} dt_1 dt_2 \cdots dt_n, \tag{2.23}$$

where the "density" $\hat{f}(t)$ is such that

$$\oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)dz_1 dz_2 \cdots dz_n = \int\limits_{-\infty}^{\infty} f(t)\hat{\phi}(t)dt_1 dt_2 \cdots dt_n \tag{2.24}$$

and the modulus of $\hat{F}(z)$ is bounded by

$$|\hat{F}(z)| \leq C|z|^p e^{[p\sum_{j=1}^{n} |\Re(z_j)|]}, \tag{2.25}$$

where C and p depend on $\hat{F}$.

3 The Case $N \to \infty$

When the number of variables of the argument of the Ultradistribution of Exponential type tends to infinity we define:

$$d\mu(x) = \frac{e^{-x^2}}{\sqrt{\pi}} dx. \tag{3.1}$$

Let $\hat{\phi}(x_1, x_2, \ldots, x_n)$ be such that:

$$\int_{-\infty}^{\infty} \cdots \int |\hat{\phi}(x_1, x_2, \ldots, x_n)|^2 d\mu_1 d\mu_2 \cdots d\mu_n < \infty, \tag{3.2}$$

where

$$d\mu_i = \frac{e^{-x_i^2}}{\sqrt{\pi}} dx_i. \tag{3.3}$$

Then by definition $\hat{\phi}(x_1, x_2, \ldots, x_n) \in L_2(\mathbb{R}^n, \mu)$ and

$$L_2(\mathbb{R}^\infty, \mu) = \bigcup_{n=1}^{\infty} L_2(\mathbb{R}^n, \mu). \tag{3.4}$$

Let $\hat{\psi}$ be given by

$$\hat{\psi}(z_1, z_2, \ldots, z_n) = \pi^{n/4} \hat{\phi}(z_1, z_2, \ldots, z_n) e^{\frac{z_1^2 + z_2^2 + \cdots + z_n^2}{2}}, \tag{3.5}$$

where $\hat{\phi} \in \mathbf{3}^n$ (the corresponding n-dimensional of $\mathbf{3}$).

Then by definition $\hat{\psi}(z_1, z_2, \ldots, z_n) \in \mathbf{\mathfrak{G}}(\mathbb{C}^n)$,

$$\mathbf{\mathfrak{G}}(\mathbb{C}^\infty) = \bigcup_{n=1}^{\infty} \mathbf{\mathfrak{G}}(\mathbb{C}^n) \tag{3.6}$$

and the dual $\mathbf{\mathfrak{G}}'(\mathbb{C}^\infty)$ given by

$$\mathbf{\mathfrak{G}}'(\mathbb{C}^\infty) = \bigcup_{n=1}^{\infty} \mathbf{\mathfrak{G}}'(\mathbb{C}^n) \tag{3.7}$$

is the space of Ultradistributions of Exponential type.

The analog to (2.11) in the infinite dimensional case is:

$$\mathbf{\mathfrak{W}} = (\mathbf{\mathfrak{G}}(\mathbb{C}^\infty), L_2(\mathbb{R}^\infty, \mu), \mathbf{\mathfrak{G}}'(\mathbb{C}^\infty)). \tag{3.8}$$

If we define:

$$\mathcal{F} : \mathbf{\mathfrak{G}}(\mathbb{C}^\infty) \to \mathbf{\mathfrak{G}}(\mathbb{C}^\infty) \tag{3.9}$$

via the Fourier transform:

$$\mathcal{F} : \mathbf{\mathfrak{G}}(\mathbb{C}^n) \to \mathbf{\mathfrak{G}}(\mathbb{C}^n) \tag{3.10}$$

given by:

$$\mathcal{F}\{\hat{\psi}\}(k) = \int\limits_{-\infty}^{\infty} \hat{\psi}(z_1, z_2, \ldots, z_n)e^{ik \cdot z + \frac{k^2}{2}} d\rho_1 d\rho_2 \cdots d\rho_n, \tag{3.11}$$

where

$$d\rho(z) = \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \tag{3.12}$$

we conclude that

$$\mathbf{\mathfrak{G}}'(\mathbb{C}^\infty) = \mathcal{F}_c\{\mathbf{\mathfrak{G}}'(\mathbb{C}^\infty)\} = \mathcal{F}\{\mathbf{\mathfrak{G}}'(\mathbb{C}^\infty)\}, \tag{3.13}$$

where in the one-dimensional case

$$\mathcal{F}_c\{\hat{\psi}\}(k) = \Theta[\Im(k)] \int\limits_{\Gamma_+} \hat{\psi}(z)e^{ikz + \frac{k^2}{2}} d\rho - \Theta[-\Im(k)] \int\limits_{\Gamma_-} \hat{\psi}(z)e^{ikz + \frac{k^2}{2}} d\rho. \tag{3.14}$$

4 The Constraints for a Bradyonic Bosonic String

We take as definition and a start point of a string theory the Nambu–Goto Lagrangian, and all developments in our theory will be based in it. As is known the Nambu–Goto Lagrangian for the bosonic string is given by [13]

$$\mathcal{L}_{NG} = T\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2}, \tag{4.1}$$

where

$$X = X(\tau, \sigma), \qquad \dot{X} = \partial_\tau X, \qquad X' = \partial_\sigma X.$$

If we use the constraint

$$(\dot{X} - X')^2 = 0 \tag{4.2}$$

we obtain:

$$\dot{X}^4 + X'^4 = 4(\dot{X} \cdot X')^2 - 2\dot{X}^2 X'^2 \geq 0. \tag{4.3}$$

On the other hand

$$(\dot{X}^2 - X'^2)^2 = \dot{X}^4 + X'^4 - 2\dot{X}^2 X'^2 \tag{4.4}$$

and from (4.3) we have

$$4\mathcal{L}_{BS}^2 = T^2(\dot{X}^2 - X'^2)^2 = 4T^2[(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2] = 4\mathcal{L}_{NG}^2 \geq 0. \tag{4.5}$$

As a consequence of (4.5):

$$\mathcal{L}_{NG} = T\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} = \frac{T}{2}|\dot{X}^2 - X'^2| = \mathcal{L}_{BS}. \tag{4.6}$$

We then see that is sufficient to use only one constraint to obtain the Lagrangian for a bosonic string theory from the Nambú-Goto Lagrangian. Another constraint from which (4.6) follows is

$$(\dot{X} + X^{'})^2 = 0. \tag{4.7}$$

Thus, the equations for the bosonic string reduce to:

$$
\begin{cases}
\ddot{X} - X^{''} = 0, \\
(\dot{X} + X^{'})^2 = 0, \\
X_\mu(\tau, 0) = X_\mu(\tau, \pi) = 0,
\end{cases}
\tag{4.8}
$$

or equivalently

$$
\begin{cases}
\ddot{X} - X^{''} = 0, \\
(\dot{X} - X^{'})^2 = 0, \\
X_\mu(\tau, 0) = X_\mu(\tau, \pi) = 0.
\end{cases}
\tag{4.9}
$$

The Euler–Lagrange equations for (4.8) and (4.9) are respectively:

$$
4\delta(\dot{X}^2 - X^{'2})[(\dot{X} \cdot \ddot{X} - X^{'} \cdot \dot{X}^{'})\dot{X}_\mu - (X^{'} \cdot \dot{X}^{'} - X^{'} \cdot X^{''})X^{'}_\mu]
$$
$$
+ \operatorname{Sgn}(\dot{X}^2 - X^{'2})(\ddot{X} - X^{''}) + \lambda(\ddot{X} + 2\dot{X}^{'} + X^{''}) = 0, \tag{4.10}
$$

$$
4\delta(\dot{X}^2 - X^{'2})[(\dot{X} \cdot \ddot{X} - X^{'} \cdot \dot{X}^{'})\dot{X}_\mu - (X^{'} \cdot \dot{X}^{'} - X^{'} \cdot X^{''})X^{'}_\mu]
$$
$$
+ \operatorname{Sgn}(\dot{X}^2 - X^{'2})(\ddot{X} - X^{''}) + \lambda(\ddot{X} - 2\dot{X}^{'} + X^{''}) = 0, \tag{4.11}
$$

where λ is a Lagrange multiplier.

If we define a physical state of the string as:

$$p^2|\Phi\rangle = 0. \tag{4.12}$$

A solution of (4.8) defined on physical states is:

$$
\begin{cases}
X_\mu(\tau, \sigma) = x_\mu + l^2 p_\mu \tau + \dfrac{il}{2} \displaystyle\sum_{n=-\infty; n\neq 0}^{\infty} \dfrac{a_n}{n} e^{-2in(\tau - \sigma)}, \\
p^2|\Phi\rangle = 0.
\end{cases}
\tag{4.13}
$$

It is immediate to prove that (4.13) is solution of Nambu–Goto equations on physical states, (Nambu–Goto equations arise from Euler–Lagrange equations corresponding to the Lagrangian (4.1), and it is easy to prove that the currently used solution for the closed string movement is not solution of Nambu–Goto equations due to the fact that Virasoro operators L_n and $\tilde{L}_n$ does not annihilate the physical states for $n < 0$) and moreover, does not form a set of commuting operators.

In a similar way a solution of (4.9) at the is:

$$
\begin{cases}
X_\mu(\tau, \sigma) = x_\mu + l^2 p_\mu \tau + \dfrac{il}{2} \displaystyle\sum_{n=-\infty; n\neq 0}^{\infty} \dfrac{\tilde{a}_n}{n} e^{-2in(\tau + \sigma)}, \\
p^2|\Phi\rangle = 0.
\end{cases}
\tag{4.14}
$$

5 The Constraints for a Tachyonic Bosonic String

The Nambu–Goto Lagrangian for the tachyonic bosonic string is given by

$$\mathcal{L}_{NG} = T\sqrt{\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2}. \tag{5.1}$$

If we use the constraint

$$(\dot{X} \pm iX')^2 = 0 \tag{5.2}$$

we obtain

$$\dot{X}^4 + X'^4 = 2\dot{X}^2 X'^2 - 4(\dot{X} \cdot X')^2 \geq 0. \tag{5.3}$$

On the other hand

$$(\dot{X}^2 + X'^2)^2 = \dot{X}^4 + X'^4 + 2\dot{X}^2 X'^2 \geq 0 \tag{5.4}$$

and from (5.3) we have

$$4\mathcal{L}_{BS}^2 = T^2(\dot{X}^2 + X'^2)^2 = 4T^2[\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2] = 4\mathcal{L}_{NG}^2 \geq 0. \tag{5.5}$$

As a consequence of (5.5):

$$\mathcal{L}_{NG} = T\sqrt{\dot{X}^2 X'^2 - (\dot{X} \cdot X')^2} = \frac{T}{2}|\dot{X}^2 + X'^2| = \mathcal{L}_{BS}. \tag{5.6}$$

6 The Propagator of the Closed Bosonic String

We write X_μ in (4.13) as:

$$X_\mu(\tau, \sigma) = x_\mu + l^2 p_\mu \tau + l \int_{-\infty}^{\infty} \left(\frac{a_\mu(k)}{\sqrt{2|k|}} e^{-i(|k|\tau - k\sigma)} + \frac{a_\mu^+(k)}{\sqrt{2|k|}} e^{i(|k|\tau - k\sigma)} \right) dk, \tag{6.1}$$

where

$$\begin{aligned} a_\mu(k) &= i \sum_{n>0} a_{\mu n} \delta(k - 2n), \\ a_\mu^+(k) &= -i \sum_{n>0} a_{\mu n}^+ \delta(k - 2n), \end{aligned} \tag{6.2}$$

with

$$[a_{\mu m}, a_{\nu n}^+] = \eta_{\mu\nu} \delta_{mn} \tag{6.3}$$

and

$$[a_\mu(k), a_\nu^+(k')] = \eta_{\mu\nu} \delta(k - k') \sum_{n>0} \delta(k - 2n) = \frac{\eta_{\mu\nu}}{4} \Theta(k) \delta(k - k') \sum_{n=-\infty}^{\infty} e^{-in\pi k}. \tag{6.4}$$

With the usual definition

$$\Delta_{\mu\nu}(\tau - \tau', \sigma - \sigma') = \langle 0|T[X_\mu(\tau, \sigma) X_\nu(\tau', \sigma')]|0\rangle$$

the propagator for the string is (do not confuse with the string field propagator of Sect. 9):

$$\Delta_{\mu\nu}(\tau - \tau', \sigma - \sigma') = \eta_{\mu\nu} \frac{l^2}{4} \sum_{n>0} n^{-1} e^{-2inSgn(\tau-\tau')[(\tau-\tau')-(\sigma-\sigma')]} \, Sgn(\tau - \tau'). \qquad (6.5)$$

For X_μ in (4.14) we obtain:

$$\Delta_{\mu\nu}(\tau - \tau', \sigma - \sigma') = \eta_{\mu\nu} \frac{l^2}{4} \sum_{n>0} n^{-1} e^{-2inSgn(\tau-\tau')[(\tau-\tau')+(\sigma-\sigma')]} \, Sgn(\tau - \tau'). \qquad (6.6)$$

7 A Representation of the States of the Closed Bosonic String

7.1 The Case n Finite

For an ultradistribution of exponential type, we can write:

$$G(k) = \oint_{\Gamma_z} \{\Theta[\Im(k)]\Theta[\Re(z)] - \Theta[-\Im(k)]\Theta[-\Re(z)]\} \hat{G}(z) e^{ikz} dz,$$

$$\hat{G}(z) = \oint_{\Gamma_k} \{\Theta[\Im(z)]\Theta[-\Re(k)] - \Theta[-\Im(z)]\Theta[\Re(k)]\} G(z) e^{-ikz} dk \qquad (7.1)$$

and

$$G(\phi) = \oint_{\Gamma_k} G(k)\phi(k) dk$$

$$= \oint_{\Gamma_k} \oint_{\Gamma_z} \{\Theta[\Im(k)]\Theta[\Re(z)] - \Theta[-\Im(k)]\Theta[-\Re(z)]\} \hat{G}(z)\phi(k) e^{ikz} dk dz \qquad (7.2)$$

$$= -i \oint_{\Gamma_k} \oint_{\Gamma_z'} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\} \hat{G}(-iz)\phi(k) e^{kz} dk dz, \quad (7.3)$$

where the path Γ_z' is the path Γ_z rotated ninety degrees counterclockwise around the origin of the complex plane.

If $F(z)$ is an UET of compact support we can define:

$$\langle \hat{F}(z), \phi(z) \rangle = \oint_{\Gamma_k} \oint_{\Gamma_z'} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\} \hat{F}(z)\phi(k) e^{kz} dk dz \quad (7.4)$$

then

$$\langle \hat{F}'(z), \phi(z) \rangle = \oint_{\Gamma_k} \oint_{\Gamma'_z} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\}\hat{F}'(z)\phi(k)e^{kz}dkdz$$

$$= -\oint_{\Gamma_k} \oint_{\Gamma'_z} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\}\hat{F}(z)k\phi(k)e^{kz}dkdz$$

$$= \langle \hat{F}(z), -z\phi(z) \rangle. \tag{7.5}$$

If we define

$$a = -z, \qquad a^+ = \frac{d}{dz} \tag{7.6}$$

we have

$$[a, a^+] = 1. \tag{7.7}$$

Thus we have a representation for creation and annihilation operators of the states of the string. The vacuum state annihilated by z_μ is the UET $\delta(z_\mu)$, and the orthonormalized states obtained by successive application of $\frac{d}{dz_\mu}$ to $\delta(z_\mu)$ are

$$F_n(z_\mu) = \frac{\delta^{(n)}(z_\mu)}{\sqrt{n!}}. \tag{7.8}$$

On the real axis:

$$\langle \hat{F}(z), \phi(z) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \overline{\hat{f}}(x)\phi(k)e^{kx}dxdk, \tag{7.9}$$

where $\overline{\hat{f}}(x)$ is given by Dirac's formula:

$$\hat{F}(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\overline{\hat{f}}(x)}{x - z}dx. \tag{7.10}$$

A general state of the string can be written as

$$\phi(x, \{z\}) = [a_0(x) + a_{\mu_1}^{i_1}(x)\partial_{i_1}^{\mu_1} + a_{\mu_1\mu_2}^{i_1 i_2}(x)\partial_{i_1}^{\mu_1}\partial_{i_2}^{\mu_2} + \cdots + \cdots$$
$$+ a_{\mu_1\mu_2\cdots\mu_n}^{i_1 i_2\cdots i_n}(x)\partial_{i_1}^{\mu_1}\partial_{i_2}^{\mu_2}\cdots\partial_{i_n}^{\mu_n} + \cdots + \cdots]\delta(\{z\}), \tag{7.11}$$

where $\{z\}$ denotes $(z_{1\mu}, z_{2\mu}, \ldots, z_{n\mu}, \ldots)$, and ϕ is a UET of compact support in the set of variables $\{z\}$. The functions $a_{\mu_1\mu_2\cdots\mu_n}^{i_1 i_2\cdots i_n}(x)$ are solutions of

$$\Box a_{\mu_1\mu_2\cdots\mu_n}^{i_1 i_2\cdots i_n}(x) = 0. \tag{7.12}$$

7.2 The Case $n \to \infty$

In this case

$$G(k) = \oint_{\Gamma_z} \{\Theta[\Im(k)]\Theta[\Re(z)] - \Theta[-\Im(k)]\Theta[-\Re(z)]\}\hat{G}(z)e^{ikz + \frac{k^2}{2} - \frac{z^2}{2}}\frac{dz}{\sqrt{2\pi}},$$

$$\hat{G}(z) = \oint_{\Gamma_k} \{\Theta[\Im(z)]\Theta[-\Re(k)] - \Theta[-\Im(z)]\Theta[\Re(k)]\}G(z)e^{-ikz+\frac{z^2}{2}-\frac{k^2}{2}}\frac{dk}{\sqrt{2\pi}}, \qquad (7.13)$$

$$G(\phi) = \oint_{\Gamma_k} G(k)\phi(k)e^{-k^2}\frac{dk}{\sqrt{\pi}}$$

$$= \oint_{\Gamma_k}\oint_{\Gamma_z} \{\Theta[\Im(k)]\Theta[\Re(z)] - \Theta[-\Im(k)]\Theta[-\Re(z)]\}\hat{G}(z)\phi(k)e^{ikz-\frac{z^2}{2}-k^2}\frac{dkdz}{\sqrt{2\pi}} \quad (7.14)$$

$$= -i\oint_{\Gamma_k}\oint_{\Gamma_z'} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\}\hat{G}(-iz)\phi(k)e^{kz+\frac{z^2}{2}-k^2}\frac{dkdz}{\sqrt{2\pi}}. $$
$$(7.15)$$

If $F(z)$ is an CUET we can define

$$\langle\hat{F}(z), \phi(z)\rangle = \oint_{\Gamma_k}\oint_{\Gamma_z'} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\}$$

$$\times \hat{F}(z)e^{-\frac{3z^2}{2}}]\phi(k)e^{kz+\frac{z^2}{2}-k^2}\frac{dkdz}{\sqrt{2\pi}} \qquad (7.16)$$

$$= \oint_{\Gamma_k}\oint_{\Gamma_z'} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\}$$

$$\times \hat{F}(z)\phi(k)e^{kz-z^2-k^2}\frac{dkdz}{\sqrt{2\pi}} \qquad (7.17)$$

and then

$$\langle -2z\hat{F}(z) + \hat{F}'(z), \phi(z)\rangle = \oint_{\Gamma_k}\oint_{\Gamma_z'} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\}$$

$$\times [-2z\hat{F}(z) + \hat{F}'(z)]\phi(k)e^{kz-z^2-k^2}\frac{dkdz}{\sqrt{2\pi}}$$

$$= -\oint_{\Gamma_k}\oint_{\Gamma_z'} \{\Theta[\Im(k)]\Theta[\Im(z)] - \Theta[-\Im(k)]\Theta[-\Im(z)]\}$$

$$\times \hat{F}(z)k\phi(k)e^{kz-z^2-k^2}\frac{dkdz}{\sqrt{2\pi}}$$

$$= \langle\hat{F}(z), -z\phi(z)\rangle. \qquad (7.18)$$

If we define:

$$a = -z, \qquad a^+ = -2z + \frac{d}{dz}, \qquad (7.19)$$

we have

$$[a, a^+] = 1. \qquad (7.20)$$

The vacuum state annihilated by a is $\delta(z)e^{z^2}$. The orthonormalized states obtained by successive application of a^+ are

$$\hat{F}_n(z) = 2^{\frac{1}{4}}\pi^{\frac{1}{2}}\frac{\delta^{(n)}(z)e^{z^2}}{\sqrt{n!}}. \tag{7.21}$$

On the real axis we have

$$\langle \hat{F}(z), \phi(z)\rangle = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\overline{\hat{f}}(x)\phi(k)e^{kx-x^2-k^2}\frac{dxdk}{\sqrt{2\pi}}, \tag{7.22}$$

where $\overline{\hat{f}}(x)$ is given by Dirac's formula:

$$\hat{F}(z) = \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\overline{\hat{f}}(x)}{x-z}dx. \tag{7.23}$$

8 The String Field

According to (4.13) and Sect. 7 the equation for the string field is given by

$$\Box \Phi(x, \{z\}) = (\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\Phi(x, \{z\}) = 0, \tag{8.1}$$

where $\{z\}$ denotes $(z_{1\mu}, z_{2\mu}, \ldots, z_{n\mu}, \ldots)$, and Φ is a CUET in the set of variables $\{z\}$. Any UET of compact support can be writen as a development of $\delta(\{z\})$ and its derivatives. Thus we have:

$$\Phi(x, \{z\}) = [A_0(x) + A_{\mu_1}^{i_1}(x)\partial_{i_1}^{\mu_1} + A_{\mu_1\mu_2}^{i_1 i_2}(x)\partial_{i_1}^{\mu_1}\partial_{i_2}^{\mu_2} + \cdots + \cdots$$
$$+ A_{\mu_1\mu_2\cdots\mu_n}^{i_1 i_2\cdots i_n}(x)\partial_{i_1}^{\mu_1}\partial_{i_2}^{\mu_2}\cdots\partial_{i_n}^{\mu_n} + \cdots + \cdots]\delta(\{z\}), \tag{8.2}$$

where the quantum fields $A_{\mu_1\mu_2\cdots\mu_n}^{i_1 i_2\cdots i_n}(x)$ are solutions of

$$\Box A_{\mu_1\mu_2\cdots\mu_n}^{i_1 i_2\cdots i_n}(x) = 0. \tag{8.3}$$

The propagator of the string field can be expressed in terms of the propagators of the component fields:

$$\Delta(x - x', \{z\}, \{z'\}) = [\Delta_0(x - x') + \Delta_{\mu_1\mu_2}^{i_1 j_1}(x - x')\partial_{i_1}^{\mu_1}\partial_{j_1}^{\prime\nu_1} + \cdots + \cdots$$
$$+ \Delta_{i_1\cdots i_n j_1\cdots j_n}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}(x - x')\partial_{\mu_1}^{i_1}\cdots\partial_{\mu_n}^{i_n}\partial_{\nu_1}^{\prime j_1}\cdots\partial_{\nu_n}^{\prime j_n} + \cdots + \cdots]$$
$$\times \delta(\{z\}, \{z'\}). \tag{8.4}$$

We define the operators of annihilation and creation of a string as:

$$a(k, \{z\}) = [a_0(k) + a_{\mu_1}^{i_1}(k)\partial_{i_1}^{\mu_1} + \cdots + \cdots + a_{\mu_1\cdots\mu_n}^{i_1\cdots i_n}\partial_{i_1}^{\mu_1}\cdots\partial_{i_n}^{\mu_n} + \cdots + \cdots]\delta(\{z\}), \tag{8.5}$$

$$a^+(k, \{z'\}) = [a_0^+(k) + a_{\nu_1}^{+j_1}(k)\partial_{j_1}^{\prime\nu_1} + \cdots + \cdots + a_{\nu_1\cdots\nu_n}^{+j_1\cdots j_n}\partial_{j_1}^{\prime\nu_1}\cdots\partial_{j_n}^{\prime\nu_n} + \cdots + \cdots]\delta(\{z'\}). \tag{8.6}$$

If we define

$$[a^{i_1\cdots i_n}_{\mu_1\cdots\mu_n}(k), a^{+j_1\cdots j_n}_{\nu_1\cdots\nu_n}(k')] = f^{i_1\cdots i_n j_1\cdots j_n}_{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}(k)\delta(k-k'),\tag{8.7}$$

the commutations relations are

$$a(k,\{z\}), a^+(k',\{z'\})] = \delta(k-k')[f_0(k) + f^{i_1 j_1}_{\mu_1\nu_1}(k)\partial^{\mu_1}_{i_1}\partial^{\nu_1}_{j_1} + \cdots + \cdots + f^{i_1\cdots i_n j_1\cdots j_n}_{\mu_1\cdots\mu_n\nu_1\cdots\nu_n}(k)$$

$$\times \partial^{\mu_1}_{i_1}\cdots\partial^{\mu_n}_{i_n}\partial^{\nu_1}_{j_1}\cdots\partial^{\nu_n}_{j_n} + \cdots + \cdots]\delta(\{z\},\{z'\}).\tag{8.8}$$

9 The Action for the String Field

9.1 The Case n Finite

The action for the free bosonic bradyonic closed string field is

$$S_{\text{free}} = \oint_{\{\Gamma_1\}}\oint_{\{\Gamma_2\}}\int_{-\infty}^{\infty}\partial_\mu\Phi(x,\{z_1\})e^{\{z_1\}\cdot\{z_2\}}\partial^\mu\Phi(x,\{z_2\})dxd\{z_1\}d\{z_2\}.\tag{9.1}$$

A possible interaction is given by

$$S_{\text{int}} = \lambda\oint_{\{\Gamma_1\}}\oint_{\{\Gamma_2\}}\oint_{\{\Gamma_3\}}\oint_{\{\Gamma_4\}}\int_{-\infty}^{\infty}\Phi(x,\{z_1\})e^{\{z_1\}\cdot\{z_2\}}\Phi(x,\{z_2\})e^{\{z_2\}\cdot\{z_3\}}\Phi(x,\{z_3\})$$

$$\times e^{\{z_3\}\cdot\{z_4\}}\Phi(x,\{z_4\})dxd\{z_1\}d\{z_2\}d\{z_3\}d\{z_4\}.\tag{9.2}$$

Both, S_{free} and S_{int} are non-local as expected.

9.2 The Case $n\to\infty$

In this case:

$$S_{\text{free}} = \oint_{\{\Gamma_1\}}\oint_{\{\Gamma_2\}}\int_{-\infty}^{\infty}\partial_\mu\Phi(x,\{z_1\})e^{\{z_1\}\cdot\{z_2\}}\partial^\mu\Phi(x,\{z_2\})dxd\{\eta_1\}d\{\eta_2\},\tag{9.3}$$

where

$$d\eta(z) = \frac{e^{-z^2}}{\sqrt{2\pi}}\tag{9.4}$$

and

$$S_{\text{int}} = \lambda\oint_{\{\Gamma_1\}}\oint_{\{\Gamma_2\}}\oint_{\{\Gamma_3\}}\oint_{\{\Gamma_4\}}\int_{-\infty}^{\infty}\Phi(x,\{z_1\})e^{\{z_1\}\cdot\{z_2\}}\Phi(x,\{z_2\})e^{\{z_2\}\cdot\{z_3\}}\Phi(x,\{z_3\})$$

$$\times e^{\{z_3\}\cdot\{z_4\}}\Phi(x,\{z_4\})dxd\{\eta_1\}d\{\eta_2\}d\{\eta_3\}d\{\eta_4\}.\tag{9.5}$$

The convolution of two propagators of the string field is

$$\hat{\Delta}(k,\{z_1\},\{z_2\}) * \hat{\Delta}(k,\{z_3\},\{z_4\}),\tag{9.6}$$

where $*$ denotes the convolution of Ultradistributions of Exponential Type on the k variable only. With the use of the result

$$\frac{1}{\rho} * \frac{1}{\rho} = -\pi^2 \ln \rho \tag{9.7}$$

($\rho = x_0^2 + x_1^2 + x_2^2 + x_3^2$ in euclidean space) and

$$\frac{1}{\rho \pm i0} * \frac{1}{\rho \pm i0} = \mp i \pi^2 \ln(\rho \pm i0) \tag{9.8}$$

($\rho = x_0^2 - x_1^2 - x_2^2 - x_3^2$ in minkowskian space), the convolution of two string field propagators is finite.

10 Discussion

We have decided to begin this paper, for the benefit of the reader, with a summary of the main characteristics of Ultradistributions of Exponential Type and their Fourier transform.

We have shown that UET are appropriate for the description in a consistent way string and string field theories. By means of a new Lagrangian for the closed string strictly equivalent to Nambu–Goto Lagrangian we have obtained a movement equation for the field of the string and solve it with the use of CUET We shown that this string field is a linear superposition of CUET. We evaluate the propagator for the string field, and calculate the convolution of two of them, taking into account that string field theory is a non-local theory of UET of an infinite number of complex variables, For practical calculations and experimental results we have given expressions that involve only a finite number of variables.

As a final remark we would like to point out that our formulae for convolutions follow from general definitions. They are not regularized expressions.

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